Microscopic noise, adaptation and survival in hostile environments

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The survival of autocatalytic agents in hostile environments depends on their ability to adapt their spatial configuration to local fluctuations. A model of diffusive reactant that extract the advantage of spatio-temporal fluctuations associated with the stochastic wandering of diffusive catalysts is discussed. Two arguments are presented for the basic processes behind this extraordinary behavior. In the first, the local colonies that evolve around any spatially advantageous region overlap in space-time and an infinite directed percolation cluster emerges. The second argument is based on the return probability of a diffusive agent that is shown to yield finite density of active "oases" with exponentially large contribution to the reactants population. The different range of applicability of these survival lower bounds to small systems is discussed.

PACS numbers: 05.40.-a, 64.60.Ak, 64.60.Ht

I. INTRODUCTION

The stochastic effects associated with microscopic noise in spatially extended reactive systems attract recently a lot of interest [1–5], with applications from the dynamics of biochemical networks to ecology, population dynamics, disease control and chemical reactions kinetics. The common analysis of these systems is based on rate equation, i.e., deterministic partial differential equations that assume large concentration of individual reactants that are believed to allow for the use of continuous variables. Crossing to an agent based, stochastic, picture one finds a variety of new effects, unseen by the "mean field" deterministic formulation. In particular, the survival (or extinction) conditions for reactants may be shifted, leading to a new type of extinction transition. It was hypothesized that, generically, these transition falls into the equivalence class of the directed percolation transition [6].

In previous publications [5, 7] the quite unexpected resilience of a system that contains autocatalytic agents and immortal catalysts has been considered, and adaptation of the autocatalytic fluctuations to diffusive noise has been shown to lead to results that differ strongly from the PDE predictions. The system contains two species: eternal catalyst agents (A) are wandering around, and diffusive reactants (B) admit internal decay mechanism and may yield an offspring only in the presence of the catalyst. It turns out that the reactants may survive even below the mean field limit due to the Poissonian fluctuations associated with the discrete nature of the catalysts. These fluctuations allow (in large enough system) for regions of positive growth rate even if the decay process is very fast. As this exponential growth is spatially correlated to the "oasis" regions, the reactants tend to concentrate around these favored regions and multiply even further. These results turn out to be of importance for wide range of applications, and few generalizations of this "AB model" has been discussed recently [8].

This striking result and its importance for understanding various systems resilience raise the question of what are the crucial assumptions leading to it. In particular the requirement for an arbitrarily large area of the surface is of relevance in real systems. In statistical mechanics terms this evokes the issue or the commutativity between infinite volume and infinite time limits. It was shown [9] that, quite generally, there is a difference between the average and the typical local concentrations of the reactants in this system, and the average may reflect rare events of zero measure, such that it remains finite, or even diverges, while the probability for survival at a point may approach zero. In view of these results, the need for a simple theory that connect the rare events into the generic scenario is obvious. This paper is devoted to the presentation of two theoretical arguments that ensure the proliferation of the reactants below the rate equation threshold and are applicable, at least at some parameter range, to small systems.

The first argument presented here is based on the plausibility assumption used to translate the AB model to a model of directed percolation. The second argument is even more radical, as it depends on the growth at a fixed location, taking into account the diffusive correlations of the catalysts.

II. THE AB MODEL

In this model [5, 7] the disagreement between the realistic stochastic process and the deterministic PDEs that pretends to describe it is emphasized for a very simple and generic system. The system includes two species: an immortal catalyst A that only diffuses randomly in space, and a reactant agents B that decay with rate δ_B and proliferate in the presence of A-s at rate $\beta_B N_A$, where N_A is the number of A agents (local density of A) at the reactant spatial location. Both A and B undergo diffusion with rates D_A and D_B respectively. Schematically, the local reactions considered are:

$$\begin{aligned} B &\longrightarrow \oslash \\ B + A &\longrightarrow B + B + A \end{aligned} \quad (1)$$

The continuum approximation of this process involves the mean-field rate equations for the densities of A and B, $a(x, t)$ and $b(x, t)$ respectively. The rate equations are:

$$\begin{aligned} \frac{\partial a(x,t)}{\partial t} &= D_A \nabla^2 a(x,t) \\ \frac{\partial b(x,t)}{\partial t} &= D_B \nabla^2 b(x,t) - (\delta_B - \beta_B a(x,t)) b(x,t), \end{aligned} \quad (2)$$

and admits very simple solution: since A only diffuses, catalyst density becomes constant μ_A (where μ_A is the average A density) after some time, and the dynamics of the reactant B is given by the linear equation,

$$b(x,t) = D_B \nabla^2 b(x,t) - M b(x,t), \quad (3)$$

where $M \equiv \delta_B - \beta_B \mu_A$ is the decay/growth rate of the system, depending on its sign. Thus the mean field theory predicts a phase transition at $M = 0$. For positive M the reactant concentration decays exponentially, while negative M yields exponential growth (proliferation). In realistic systems, of course, one expects some saturation mechanism that prevents explosion, perhaps in the form of B agents competition for resources, but for the sake of simplicity, in the present work we neglect such a term, so the system admits two fixed points, $b = 0$ and $b = \infty$.

The unexpected resilience of the B population in the agent based version of the AB model can be traced to the following mechanisms. Any arbitrary B death rate (leading to arbitrary local exponential decay) can be balanced by the growth induced by a large enough A aggregate at the same location. This effect could in principle be washed away by the fact that, due to A diffusion, the large A aggregates are rare (their spatial density decays exponentially with their size) and short lived (the probability for an A to remain a time t at a give location decays with t). However there are 3 main effects that come to the rescue of the B population survival:

1. Even if an A aggregate decays, there is a certain probability that by the time of its decay another A aggregate will arise in its neighborhood. This will ensure the descendance of the B's that the first aggregate generated.

2. The exponential growth of the B population around large enough A aggregates compensates in the growth expectation the exponential decay of the aggregate survival probability.

3. In one or two dimensions, the probability for an A that left the aggregate to eventually return is 1.

Each of these processes is affected differently by the finite size. In the following sections we will study each of them separately. We will first address and quantify the first effect, and then address the last two.

We review here shortly the main features of the AB model for discrete agents [5, 7]. On each site x there exists at any time t an arbitrary number $N_A(x, t)$ of particles of type A and an arbitrary number $N_B(x, t)$ of particles of type B. Given an initial configuration $\{N_A(x, 0), N_B(x, 0)\}$ one generates the subsequent configurations $\{N_A(x, t), N_B(x, t)\}$ iteratively according to the following rules:

1. The particles A never "die" or "get born". They can jump on any of the d-dimensional sites neighboring their current location with a hopping probability of $D_A/2d$ for each neighbor.

2. The particles B can jump to any of the neighbors, with a hopping probability of $D_B/2d$.

3. The B's "die" with probability per unit time δ_B.

4. Any pair of A and B located on the same lattice site, can generate a new B with probability per unit time β_B.

In the following, some of the argumentation deals also with the generalized AB model, where A creation-annihilation processes are also allowed, keeping μ_A constant. The particular form of these reactions is $A \to \emptyset$ at rate δ_A and $\emptyset \to A$ at rate β_A, such that $\mu_a = \beta_A/\delta_A$.

III. PROLIFERATION BASED ON STATIC CATALYSTS

As noted in [7] and [9] in case of static catalysts ($D_A = 0$) proliferation always takes place for an infinite size system. Random distribution of the catalysts A implies spatial Poissonian fluctuations of the local growth rate, and this, in turn, implies that for large enough samples there will be finite density of "oases" (spatial regions where the overall growth rate is positive) no matter how large δ_B is. If there is no dynamics for the catalysts at least a finite fraction of these sites became "active" (i.e., yield a flourishing colony of reactants), resulting in the proliferation phase.

A. Single oasis and colony dynamics

The case of local growth of diffusive reactants on spatial domain with a single active site may be solved exactly and yields a simple and intuitive framework to be used in the following. For the sake of simplicity we present here an off lattice evaluation with some assumptions about the shape and size of the active site, but a possible generalizations for the other cases is straightforward and the basic intuition is the same. At the end of this subsection the translation of the results to discrete lattice is presented.

The growth around an active site is described by the equation:

$$\frac{\partial b(\mathbf{x},t)}{\partial t} = \widetilde{D}_B \nabla^2 b(\mathbf{x},t) + g(\mathbf{x}) b \qquad (4)$$

where $g(x)$ is a parameter that incorporates all the parameters that effects the growth, $g(x) = \beta_B N_A(x) - \delta_B$. A single active site of radius R corresponds to:

$$g(r) = \begin{cases} g_0 & r < R \\ -g_1 & r > R \end{cases} \qquad (5)$$

where g_0 and g_1 are positive. This linear problem may be solved (see [8], appendix A) using spherically symmetric functions in the physical dimensions.

Eq. (4) is linear and its solution involves the presentation of a complete set of eigenfunctions $\phi_n(r)$ with the corresponding eigenvalues Γ_n. Accordingly, any initial state $b(x,0)$ may by written as a superposition of the eigenfunction $b(x,0) = \sum_n \alpha_n \phi_n$ and its time evolution is given by

$$b(x,t) = \sum_n \alpha_n \phi_n e^{\Gamma_n t}. \qquad (6)$$

Solving (4), one finds that there are two types of eigenfunctions. Localized eigenfunctions decay exponentially out of the "oasis", $\phi_n(r) \sim exp(-\kappa_n r)$, and admit positive eigenvalues, while extended eigenvalues has zero support on the oasis and allow only negative Γ. The profile of a positive eigenfunction is thus,

$$\phi_n(r,t) \sim e^{-\kappa r + \Gamma_n t}. \qquad (7)$$

so a level point (a point of constant hight $\phi(r,t) = \text{const}$ on the profile) travels away from the oasis with velocity $v = \Gamma/\kappa$. If the B agents dynamics is approximated as a continuum dynamics [like in (4)] with finite *threshold*, this threshold dictates the level point discussed above, but as long as the concentration of the reactants on the oasis is large (compared to unity) the velocity is threshold independent. Clearly our interest is not in all the spectrum of the linearized evolution operator but only in its fastest growing (maximal Γ) state.

Let us consider now the details of the fastest growing state in the physical dimensions. First we note that in 1d there is *always* a localized solution with positive eigenvalue, independent of the reactants diffusion constant, while in three dimensions this is not the case, and for any oasis there is no bound state if the reactant diffusion is large. In 2d the situation is marginal, since this is the critical dimension for the return of a random walker: there is a positive eigenvalue for each oasis, independent of the reactant diffusion rate, but for very weak island ($g_0 R^2/\widetilde{D}_B << 1$) the growth rate $\Gamma = \widetilde{D}_B exp(-4\widetilde{D}_B/g_0 R^2)/R^2$ approaches zero exponentially as diffusion grows. Another peculiarity of the d=2 case is logarithmic corrections to the exponential decay of the spatial profile far away from the oasis and the corresponding correction to the velocity of the colony.

In the limit of "strong" oasis ($g_0 R^2/\widetilde{D}_B >> 1$) it is possible to write down the asymptotic velocity of the colony's front as:

$$v = sqrt[\widetilde{D}_B(g_0 - \theta^2 \widetilde{D}_B/R^2)] \qquad (8)$$

where $\theta = \pi/2, [z_0^{(1)}]$ and π in one, two and three dimensions correspondingly ($[z_0^{(1)}] = 2.402...$ is the first zero of the zero order Bessel function).

The translation of the above results to the discrete lattice dynamics is trivial, as the basic length scale of is now the lattice constant l_0. The diffusion coefficient of the lattice is just a hopping rate and is related to the continuum diffusion by $D_B \equiv \widetilde{D}_B/l_0^2$. The velocity, again, is given by the continuum velocity multiplied by l_0, where D_B should be plugged instead of $\widetilde{D}_B$. The radius of the oasis is of order l_0, but a better approximation for the coefficient is $\theta^2 \sim 2d$.

B. Finite size effects for autocatalytic growth on heterogenous environment

Following the analysis of the single oasis problem, let us discuss, still for immobile catalysts, the effects of finite sample size and sample to sample fluctuations, where in this subsection a lattice of N spatial sites is considered. As shown in the last section, if the number (m) of catalysts at a point is larger than some critical number m_c (that depends on the dimensionality of the system and the reactants death rate) there is an oasis and a B colony grows. As the A-s are randomly distributed, the probability to find m catalysts at a lattice point obeys the Poissonian distribution:

$$P(N_A(x,t) = m) = \frac{e^{\mu_A} {\mu_A}^m}{m!}. \qquad (9)$$

so the probability for an active oasis is the sum over (9) from m_c to infinity:

$$P(m > m_c) = 1 - \frac{\Gamma(m_c, \mu_A)}{\Gamma(m_c)}. \qquad (10)$$

If the number of lattice sites is much larger than $1/P(m > m_c)$ one expects a finite density of active sites in each random sample, while if the number of lattice sites is much smaller than $1/P(m > m_c)$ only (exponentially) rare samples will be "active" (i.e., contain at least one oasis) while all the others are inactive, and the typical case (determined by the inactive samples) differ from the average (determined by exponentially rare fluctuations). In terms of the lattice constant l_0 the typical distance between oases is $R \sim l_0(P(m > m_c))^{-d}$, and if the probability to find an active site is small, $P(m > m_c)$ may be approximated by P_{m_c}.

IV. DIRECTED PERCOLATION AND THE PROLIFERATION PHASE

As already discussed in [7, 9], a large enough sample with frozen catalyst always support the proliferation of the reactants as there is no bound, in this model, for the growth of a colony based on a single oasis. The case of diffusive catalysts is different: here the oasis is unstable, since any individual local fluctuation of the A concentration should decay in time. The main question here is the lifetime of a fluctuation. Given μ_A, the average number of catalysts per site, and a fluctuation "hight" $m > m_c$ what is the typical time until m decays to m_c. In case of diffusion of continuous field with random initial conditions, this problem is known as the problem of persistence diffusion, and the probability of a fluctuation to maintain its sign (i.e., that it never crosses the average) until t has been shown to obey a power law distribution, $P(t) \sim t^{-\theta}$, where θ depends upon the dimensionality of the system [10, 11]. This power law behavior has been demonstrated also for the probability for no crossing of arbitrary values that differ from the average, with different θ [12]. However, the problem considered here differs from these persistent diffusion cases as it involves the stochastic wandering of *discrete* agents (the A-s). Our numerical results (to be published elsewhere) indicate that the stochastic noise leads to crossover from power law to *exponential* decay, at least for the relevant time scales. This crossover will be discussed elsewhere, but for the sake of this work let us try to find a lower bound to the persistence of the oasis. This lower bound is simply the inverse hopping rate $1/mD_A$, since in the worst case any catalyst that leaves the oasis never returns so the diffusion plays the same role as death rate for the A-s in the active site. This idea may be extended to include other processes in the catalysts dynamics; for example, if the A-s decay with probability δ_A the lower bound is $1/(mD_A + m\delta_A)$ and so on.

A. From single oasis to directed percolation

At this point we gather all the information from the above sections in order to build a directed percolation picture for the AB extinction transition. In the limit of infinite sample we already have a finite density, $1/P(m > m_c)$, of active sites, and the lifetime of such an active site is approximated by a constant $1/mD_A$. In a space time diagram the active sites may thus be approximated as "rods" of constant length. Around each of these rods a B-colony is developed, with a size that grow linearly in time, so these colonies (in 1+1 dimensions, for example) looks like a triangles centered around each of these rods.

Within this picture, the problem of extinction-proliferation transition is translated into the problem of directed percolation of the space time triangles (in 1+1, or cones in 2+1 etc.). The density of the triangles is determined by the A agents statistics, their length (timespan) by the catalysts dynamics, the width of the triangles is related to the reactants growth and diffusion rate.

As described above, given the parameters δ_B, μ_A, β_B and the hopping rate for each of the species, a site is active if $N_A(x) > \delta_B/\beta_B$ (as explained above, in $2d$ this is enough condition for the existence of an oasis, where colony may grow). A lower bound for the lifetime of such an island is $\beta_B/[\delta_B(\delta_A + D_A)$. We now take the limit of "strong" oasis, (8), in order to get an order of magnitude estimation given that the number of catalysts needed at a single spatial point is not large, so fluctuations of $O(1)$ correspond to the strong oasis limit, and assume that $g_0 - 2dD_B$ is, again, of order one. If this is the case the velocity of the colony front is $\sqrt{D_B\beta_B l_0^2}$, and thus the radius of the colony when the oasis disintegrated is $R_0 = \beta_B\sqrt{D_B\beta_B l_0^2}/[\delta_B(\delta_A + D_A)$. On the other hands, the typical distance between neighboring oases scales like the inverse probability, and in 2d for rare oases this will be $R_1 = \exp[-\delta_B/(2\beta_B\mu_A)]$. The feeling fraction of the sample is given by the ration $p = (R_0/R_1)^d$, and the fate of the system is determined by the relation between p and p_c^d, the critical filling fraction for directed percolation at certain dimensionality.

One important exception is the case of single A based proliferation, i.e., where a site is active (admits local growth) if it contains a single catalyst. In this case the considerations presented above regarding the lifetime of an oasis are irrelevant, since there is strong correlation between the spatial location of different oases (in fact, the next active point appears in an adjusting site due to the diffusive wandering of the catalyst). This phenomena manifest itself in Figure 1: at the single A regime the extinction transition is almost independent of the system size (the critical size grow linearly with the inverse density of the catalysts). In the opposite regime, i.e., where an active site needs large number of catalyst, the corresponding fluctuations are exponentially rare, implying strong dependence on the system size. In the next section, this single A proliferation based limit will be considered in detail.

V. LOCAL CRITERIA FOR POPULATION GROWTH

In this section we present a different root for B proliferation below the mean field threshold, based on local considerations and the return probability of a diffusive particle. To begin, let us concentrate on a fixed spatial site. The reactants growth at this site depends on the local A population, and, since our interest is in a lower bound, we assume that any B agent that leaves this site diffusively never returns and just disappears from the system. As for the catalyst dynamics, we neglect again the activity at neighboring sites and compress all the information about the diffusion of catalysts into one quantity, the function $Q_d(t)$, the probability of a diffusive particle, leaving the origin at $t = 0$, to return to the origin be-

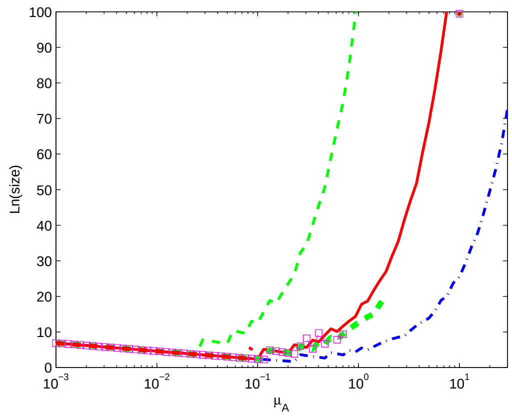

FIG. 1: Effect of scale . The mean field description of the AB model (either with or without catalyst death) is only a function of $M = \delta_B - \mu_A \beta_B$. These systems perform very differently in the strong (low values of μ_A and high values of β_B) and weak (high values of μ_A and low values of β_B) coupling limits. A clear difference is the effect if the system size. Each curve represents the system size required for survival for a given value of M ($\delta_B = 1.0$ and $\beta_B \mu_A$ are 0.05, 0.2 and 0.5 in the long dashed, full and dashed-doted line respectively. At high values of μ_A, large system is required in order to find a fluctuation large enough to allow survival. At low values of μ_A, only single A agent is required to allow survival.

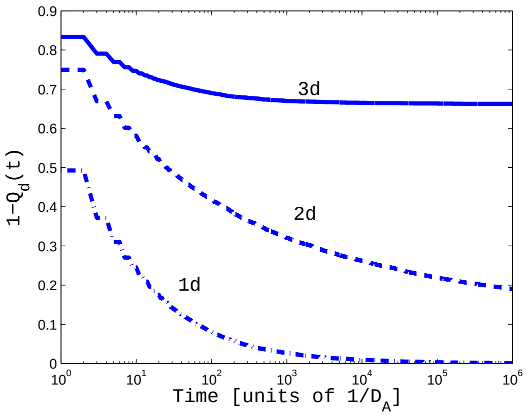

FIG. 2: A dynamics in 1, 2 and 3 dimensions. The drawn curves represents one minus the probability that A agents diffusing from a given point would return to this point at least once within a period of **t**. The probability of not returning converges rapidly to zero in one dimension, more slowly in two dimensions, and converges to a finite values in three and more dimensions.

fore time t. The limit $Q_d(t \to \infty)$, i.e., the probability to return at all is known as the Polya number and takes the value 1 for dimensions one and two, while at $d = 3$ Polya number is about 0.34. A graph representing $Q_2(t)$ is shown in Fig. 2.

One sees that while the asymptotic behavior of $Q_2(t) - 1$ is known to be $\sim 1/lnt$, in practice $Q_2(t) \sim 1$ after a very short time.

Let us calculate first the expected number of reactants $E(N_B[0,t])$ due a catalyst whose initial location is, say, the origin. The probability that the A agent stays at the origin a time t and then leaves within the interval $[t, t+\Delta t]$ is

$$exp(-D_A t) D_A dt \quad (11)$$

The B increase factor associated with such an event is $exp(\beta_B t)$. The contribution to the expected B-growth factor is the product of these two quantities:

$$exp[(\beta_B - D_A)t] D_A dt. \quad (12)$$

Let us consider separately the cases $\beta_B > D_A$ and $\beta_B < D_A$.

For the $\beta_B > D_A$ case, since the decrease in the B population due to catalyst death and migration is $exp(-[\delta_B + D_B]t)$, the condition for increase of the expected reactant population at this site is that the initial number of A's at that site N_A is larger than:

$$N_A > N_0 = (\delta_B + D_B)/(\beta_B - D_A) \quad (13)$$

For such points x where (13) holds, the expected B occupancy will grow faster than:

$$E_{t\to\infty}(B(x,t)) > B(x,0)\exp[t(N_A - N_0)(\delta_B + D_B)] \quad (14)$$

But on an initial Poisson distribution there is a finite density $\rho(N_A)$ of such points [given by Equation (10 with $m_c = N_A$]. Under these conditions the average B-population $\bar{B}(t)$ for any A-configuration history (up to measure zero) is growing exponentially:

$$\bar{B}(t)_{t\to\infty} > \bar{B}(0)\rho(N_A)\exp[t(N_A - N_0)(\delta_B + D_B)] \quad (15)$$

Incidentally, a corollary of this result is that even in the presence of death rate δ_A for the catalysts (without being replaced) the B population grows whenever $\beta_B > D_A + \delta_A$. Another observation is that the origin of time is arbitrary and any site which at *any time* fulfills (13) contributes an exponentially growing expected factor to the average B population. In particular, sites that are in a finite neighborhood of sites that fulfilled in the initial configuration (or at some stage) (13) have a finite probability to fulfill it eventually and thus contribute too an exponentially increasing expected factor to the total B population. Thus while the nature of the mathematical argument is ostensibly fixed location, the actual dynamics described by it may consist of slowly diffusing centers of growth.

For $\beta_B < D_A$, one has to preform a finer analysis which will show that the survival condition is:

$$\beta_B/D_A > 1 - Q_d(\infty). \quad (16)$$

(*In particular for* $d = 1$ *and* $d = 2$ *the survival and growth of the B-population is insured for any finite value of the parameters since* $Q_1(\infty) = Q_2(\infty) = 1$).

In order to prove (16), one has first to estimate the total expected growth factor of the B population produced by a single A during its initial stay at its point of origin. This is given by the integral of (12) over t:

$$F_1 = \frac{1}{(1-\beta_B/D_A)}. \quad (17)$$

Then, one has to estimate the effect of multiple returns of a single A (ignore for the moment multiple A initial occupation, B death and B emigration).

Consider a time τ such that the return probability before τ is $Q(\tau) > (1-\beta_B/D_A)$ (this is always possible if (16) holds). In fact given the shape of the graph in Fig. 2, this typically happens for relatively modest values of τ. Then the probability of n returns before time $n\tau$ is at least $[Q_d(\tau)]^n$. On the other hand, the B-s growth associated with such an event is about ${F_1}^n = 1/(1-\beta_B/D_A)^n$. Therefore the expected growth factor due to n returns before $t = n\tau$ is at least $[Q_d(\tau)/(1-\beta_B/D_A)]^{(t/\tau)}$, i.e., the expected contribution to B growth due to multiple returns by a single A is at least $\exp(\eta t)$ with some $\eta > (ln[Q_d(\tau)/(1-\beta_B/D_A)])/\tau$. This is an important result as it shows that any catalyst contributes an exponential time growth to the expected proliferation of the B population on its initial site. Thus the expected exponential decrease in the B population due to migration and death, $exp(-[\delta_B + D_B]t)$ at a point is balanced if the initial number of catalysts N_A at this point exceeded the critical value:

$$N_A > N_0 = (\delta_B + D_B)/\eta. \quad (18)$$

For such points x , the expected B population is guaranteed to grow faster then

$$E_{t\to\infty}(B(x,t)) > B(x,0)\exp[t(N_A - N_0)\eta] \quad (19)$$

Again, there is, at any time, a finite density $\rho(N_A)$ (given by (10)) of such points. Therefore the *expected* value $E[B(x,t)]$ at (19) insures an exponential growth in the *average* B-population $\bar{B}(t)$:

$$\bar{B}(t)_{t\to\infty} > \bar{B}(0)\rho(N_A)\exp[t(N_A - N_0)\eta] \quad (20)$$

The above estimations were made for fixed sites x where the *initial* occupancy $N_A(x,0) > N_0$. Thus, the mechanism is independent of the spatial properties of the reactants B, and, in particular, on the space-time overlap among the B colonies. Instead, it depends only on the A dynamics i.e. Poissonian fluctuations and the Polya persistence property.

Moreover, in practice the Polya property implies that the sites neighboring x have a high persistence of A occupancy too (in fact have probability 1 to be eventually visited by any A's that visited x in the past) and thus have a time increasing expected B-occupancy. Thus this mechanism extends the single-A approximation to conditions in which the appearance of independent oasis is not probable enough to sustain life. Moreover it insures life even in conditions in which the typical distance of a new active site to the original oasis [as implied by Equation (10)]is too large compared with typical colony size. In these cases, the island motion is very slow and its survival is connected to the high return probability of a diffusive agent A which insures that a large fluctuation in a certain region is preserved for much longer time then naively expected.

A similar analysis insures that for an infinite size system, even in the presence of systematic death of the catalyst δ_A (*without* A re-introduction) life prevails if

$$\beta_B > D_A[1 - Q_d(\infty)] + \delta_A. \quad (21)$$

However one should be aware of a caveat in (21). While the arguments generalizing the $single - A$ approximation (percolation, Polya-persistence) are relevant for finite size systems (as long as the size is larger then $1/P_0$ (10)), the results of the type (21) have a weaker (yet more fascinating) status. They relay on the fact that certain chains of events lead to an exponential increase in the B-population. Thus even if the probability for such a chain of events to continue for a time t decreases exponentially, it may still imply an exponential growth in the expected B-population.

Note that in order to convert the fixed site *expectation* growing result (19) into an actual *average* growing result of the type (20) any one of the following two alternatives is sufficient:

1. averaging over all A-configuration histories (at fixed site)

2. averaging over the sites of the entire system in the infinite system limit for a fixed catalysts history.

Thus, if one asks: what is the fate in statistical average (over all the stochastic realizations of the AB process) of the B occupancy at a particular fixed point? The answer is "always growing". If one asks: what is the fate of the average B-population in a fixed realization of the A-diffusion process? The answer is again: the B-population is growing in time as (20) for each fixed realization (up to measure zero). Yet , as observed in [9] if one asks: what is the fate of the B's at a fixed site for a fixed stochastic process realization? The answer may be in certain regions of the parameter space: (almost) certain extinction.

To understand the implications of such a scenario imagine and infinite universe populated by autocatalytic entities [13]. At some stage there might be a certain density of civilizations active (disconnected islands). Life could survive in such an universe in a few modes. One mode, suggested by the "single-A" approximation (reinforced by Polya persistence arguments above) is that each civilization survives independently by slowly shifting to adapt to the changes in its environment (jumping from one planet to another etc). Another mode, suggested by the directed percolation argument is that upon each cataclysm that destroys the center of a civilization , there are typically fringe colonies that are just able (or lucky) enough to find local resources for their independent survival and for recreating a (re-)new(-ed) civilization. Yet another possibility is that in fact at some stage in the history of the universe the conditions become increasingly impossible for life. As explained above this may lead to an exponential decrease in the probability of survival at each particular location. Still, the exponential growth at the locations where life survives is enough to insure that the total B-population in the universe increases continuously.

Thus one approaches asymptotically a paradoxical state in which measure zero of the system's volume supports an ever increasing amount of life. In fact this argument is not irrelevant for finite systems: they will have a finite survival time but - due to the rare spots with exponentially growing population - this time will be much longer then predicted by the naive continuous (rate equation) based estimations. Of course the argument also holds for species dynamics in the real or in the genome space: life can survive in apparently hopeless conditions either by species adaptively drifting form one location to another, or by having fringe individuals surviving cataclysms and starting new species, or by simply disappearing at the bad locations while exploiting exponentially the rare propitious spots. The actual predictions that these scenarios make for specific populations in various finite space and time conditions should be confronted with actual empirical observations in population biology or even human social and cultural history.

From the above discussion it follows that, in the thermodynamic limit (infinite system size) in 1 and 2 dimensions, the system is always in the "life" phase. Yet the way this happens is very non-trivial if one chooses arbitrarily life-hostile parameters. Indeed in the conditions of arbitrarily large death rate and arbitrarily small birth rate, etc. the set of sites where life survives may decrease exponentially. In this range of parameters, if one measures the B occupation at a *fixed* point (say, the origin), the probability to find reactant population at this site approaches zero, as shown by Kesten and Sidoravicius [9]. However, under the above conditions the spatial distribution is very singular and the results about a single fixed point do not reflect the overall population of the system. In fact the exponential growth of the B-population at those locations where life survives is enough to insure that in the thermodynamic sense (average of the B-occupancy over all the system sites in the infinite size limit) the B population always increases for all (except of a zero measure of the set of) A-configuration histories. Thus in spite of the result in [9], according to the common definitions of a phase transition, there is no phase transition in this system for d=1 and d=2 because the system is always in the growing phase.

VI. CONCLUSIONS

In addition to the existentialist worries about the fate of the human race or one's own civilization / descendence there are more concrete applications of the present work. The current rapidly shrinking natural habitats pose an immediate danger to the very existence of some species. The same mathematical formalism represents also the problems related to the sustainability of the present world economic system [14]. There, the effects of the market size and globalization are presently under close scrutiny and heated debate. We here propose a novel mathematical treatment of the habitat / economy size effects on system resilience and sustainability. This treatment is based on fluctuation induced growth. In general terms, that the dominating mechanism is that rare fluctuations affect the survival probabilities of species / economies under usual and very hostile conditions. In the present paper we have shown how does the survivability of the population depend on the system size. We have shown that as a function of the catalyst (food, prey, inventions, know-how) properties, one can obtain unlimited growth, limited growth, total extinction, or an increasingly singular distribution of growth. A meta analysis could provide in the future the survival probabilities in specific systems in the presence of various constraint: for instance compare living habitats consisting of disconnected patches with very large systems where the size effects are negligible. We hope in future publications to present on-going studies that confront the present theoretical results with empirical data from markets [15] economics [14] population dynamics and other branches of science.

VII. ACKNOWLEDGEMENTS

This work was supported by the Israeli Science Foundation (grant no. 281/03). YL and NS thank also the Yeshaya Horowitz Association through The Center for Complexity Science.

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